Prep 3

October 16, 2017 10:24 AM

Math club Problem Set #3 Conics:

1. Find the centre and radius of the circle: $4x^2+4y^2+12x+16y+9=0$

$$4x^2+12x+4y^2+16y+9=0$$
$$4(x^2+3x)+4(y^2+4y)+9=0$$
$$4\left(x^2+3x+\tfrac{9}{4}\right)-9+4(y^2+4y+4)-16+9=0$$
$$4\left(x+\tfrac{3}{2}\right)^2+4(y+2)^2=16$$
$$\left(x+\tfrac{3}{2}\right)^2+(y+2)^2=4.$$

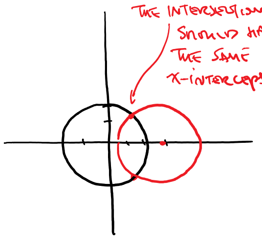
CENTRE $\left(-\frac{3}{2},-2\right)$ RADIUS $=2$ //

2. A chord of a circle is a line segment whose endpoints are on the circle. Find the length of the common chord of the two circles whose equations are $x^2+y^2=4$ and $x^2+y^2-6x+2=0$

① NOTE: FIND THE INTERSECTION PTS OF THE TWO CIRCLES

$$x^2-6x+y^2+2=0$$
$$(x^2-6x+9)-9+y^2+2=0$$
$$(x+3)^2+y^2=7.$$

THE INTERSECTIONS SHOULD HAVE THE SAME x-INTERCEPTS



② SUBSTITUTE THE EQUATIONS:

$$\underbrace{x^2+y^2}-6x+2=0$$
$$4-6x+2=0$$
$$6=6x$$
$$\boxed{1=x}\ //$$

③ FIND THE y-VALUES.

$$1^2+y^2=4$$
$$y^2=3$$
$$y=\sqrt{3}$$

LENGTH $=2y=2\sqrt{3}$.

3. Find the area enclosed in the graph of $x^2+y^2=16x+32y$

① COMPLETE THE SQUARE + GET THE RADIUS.

$$x^2-16x+y^2-32y=0$$
$$(x^2-16x+64)-64+(y^2-32y+256)-256=0$$
$$(x-8)^2+(y-16)^2=320$$

$$\begin{array}{r} {}^{1}2{}^{1}56 \\ 64 \\ \hline 20 \end{array}$$

$$R^2=320$$

② AREA $=\pi R^2$

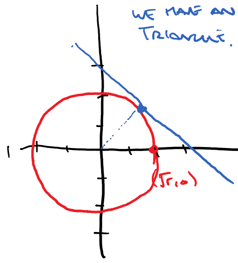
$=320\pi$ units^2 //

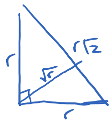
4. Find "r" if "r" is positive and the line whose equation is $x+y=r$ is tangent to the circle whose equation is $x^2+y^2=r$ [AHSME]

① TANGENT MEANS IT TOUCHES AT ONLY ONE PT.

② $x+y=r \rightarrow y=-x+r$.

WE MAKE AN ISOSCELES RIGHT TRIANGLE.





$r^2 = r\sqrt{r}(\sqrt{2})$

$r = \sqrt{r}(\sqrt{2})$

$\frac{r}{\sqrt{r}} = \sqrt{r} = \sqrt{2}$

$\boxed{r=2}$

③ ALTERNATE METHOD (SUBSTITUTION)

$x^2+y^2=r$

$x^2+(-x+r)^2=r$

$x^2+x^2-2xr+r^2=r$

$2x^2+2xr+r^2-r=0$

$a=2 \quad b=2r \quad c=r^2-r$

$b^2-4ac=0$ (ONLY ONE soln)

$4r^2-4(2)(r^2-r)=0$

$4r^2-8r^2+8r=0$

$0=4r^2-8r$

$0=4r(r-2)$

$r=0$, or $\boxed{r=2}$

NOPE

5. An airline charges \$200 per ticket and sells 40,000 tickets. For every \$10 increase they sell 1000 fewer tickets. How much should they charge to maximize their revenue?

① $\frac{Q-40000}{P-200} = \frac{-1000}{10}$

$Q-40{,}000 = -100(p-200)$

$Q = -100p + 20000 + 40{,}000$

$\boxed{Q=-100p+60{,}000}$

② $R = p \times Q$

$= p(-100p+60{,}000)$

$p=0 \qquad p=600$

③ AVG of x-int GIVES vertex (x value)

$\frac{0+600}{2} = \underline{\underline{\$300}}$

6. Find the smallest possible value of the length of a diagonal of a rectangle with perimeter 36. Prove that your answer is the shortest possible diagonal. Do not just provide an answer:

① $l+w=18 \rightarrow l=18-w$

② $D^2 = l^2+w^2$

$= (18-w)^2+w^2$

$= 18^2-36w+2w^2$

$= 2w^2-18w+162$

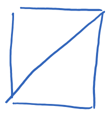
$D^2 = (w^2-18w+81)-81+162$

$D^2 = (w-9)^2+81$

$18 = 9 \times 2$

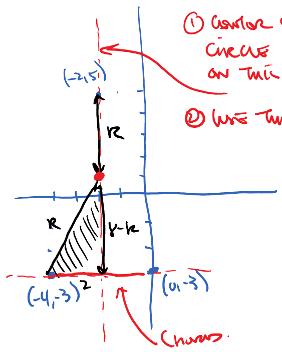
9×4

81
$\times 4$
$\overline{320}$



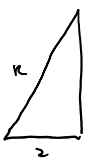
9×2
$9 \times 2 = 16$

7. The circumcircle of a triangle is the circle that passes through all three vertices of the triangle. Find an equation whose graph is the circumcircle of a triangle with vertices $(-2,5)$, $(-4,-3)$, *and* $(0,-3)$



① CENTER OF THE CIRCLE IS SOMEWHERE ON THIS LINE

② USE THE PYTH. THM.



$2^2+(8-R)^2=R^2$

$4+64-16R+R^2=R^2$

$68=16R$

$\frac{68}{16}=R.$

$\boxed{\frac{17}{4}=R}$

③ FIND CENTER:

$5-\frac{17}{4}=\frac{3}{4}$ (y-value)

$x=-2$ (x-value)

④ EQN OF CIRCLE:

$\boxed{(x+2)^2+\left(y-\frac{3}{4}\right)^2=\frac{289}{16}}$

8. Points "A" and "B" are on the parabola $y=4x^2+7x-1$, and the origin is the midpoint of $\overline{AB}$. What is the length of $\overline{AB}$?

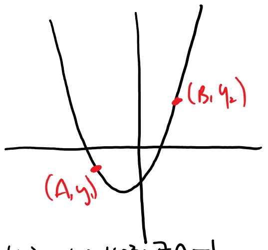
① LET 'A' & 'B' BE THE X-VALUES ② DEFN OF MID PT.

③ $y_A=4\left(\frac{1}{2}\right)^2+7\left(\frac{1}{2}\right)-1$

$=1+3.5-1=3.5$

length of $\overline{AB}$?

① LET 'A' & 'B' BE THE X-COORDS



$p+(A): \ y = 4A^2 + 7A - 1$

$p+(B): \ y = 4B^2 + 7B - 1$

② DEFN OF MID PT.

$\frac{A+B}{2} = 0 \rightarrow \boxed{A = -B}$

$\frac{4A^2 + 7A - 1 + 4B^2 + 7B - 1}{2} = 0$

$4(A^2) + 4(B^2) + 7A + 7B - 2 = 0$

$4(A^2) + 4A^2 + \cancel{7A} + \cancel{7(-A)} - 2 = 0$

$8A^2 = 2$

$A^2 = 1/4$

$\boxed{A = 1/2}$ $\boxed{B = -1/2}$

③ $y_A = 4(\frac{1}{2})^2 + 7(\frac{1}{2}) - 1$

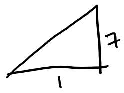
$= 1 + 3.5 - 1 = 3.5$

$y_B = 4(-\frac{1}{2})^2 + 7(-\frac{1}{2}) - 1$

$= 1 - 3.5 - 1$

$= -3.5.$

④ LENGTH $\overline{AB}$:



$7^2 + 1^2 = 50$

$\boxed{D = \sqrt{50}}$

9. Let $f(x) = x^2 + 6x + 1$, and let "R" denote the set of points (x,y) in the coordinate plane such that $f(x) + f(y) \le 0$ and $f(x) - f(y) \le 0$. Find the area of "R" [AMC]

$f(x) = x^2 + 6x + 1$

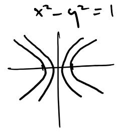
$= (x^2 + 3x + 9) - 8.$

$= (x + \frac{3}{2})^2 - 8.$

$f(y) = y^2 + 6y + 1$

$= (y + \frac{3}{2})^2 - 8$

$x^2 - y^2 = 1$



$f(x) + f(y) \le 0.$

$(x + \frac{3}{2})^2 - 8 + (y + \frac{3}{2})^2 - 8 \le 0$

$(x + \frac{3}{2})^2 + (y + \frac{3}{2})^2 \le 16.$

RADIUS = 4.

$\boxed{\text{AREA} = 16\pi}$

$f(x) - f(y) \le 0$

$(x + \frac{3}{2})^2 - 8 - (y + \frac{3}{2})^2 + 8 \le 0$

$(x^2 + \frac{3}{2})^2 - (y + \frac{3}{2})^2 \le 0.$

[HYPERBOLA] $\boxed{\text{AREA} = 0}$

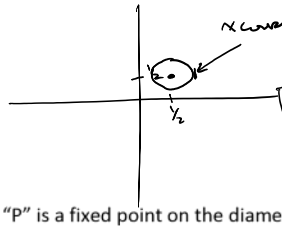
10. Find the largest value of "x" for which $x^2 + y^2 = x + y$ has a solution, given that "x" and "y" are real. [ARML]

$x^2 - x + y^2 - y = 0$

$(x^2 - x + \frac{1}{4}) - \frac{1}{4} + (y^2 - y + \frac{1}{4}) - \frac{1}{4} = 0$

$(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = \frac{1}{2}$

THIS IS A CIRCLE WITH A RADIUS of $\frac{1}{\sqrt{2}}$



x COORD $= \frac{1}{2} + \frac{1}{\sqrt{2}}$

$= \frac{1}{2} + \frac{\sqrt{2}}{2}$

$\boxed{x = \frac{1+\sqrt{2}}{2}}$

CHECK:

$\left(\frac{1+\sqrt{2}}{2}\right)^2 + \frac{1}{4} = \frac{1+\sqrt{2}}{2} + \frac{1}{2}$

$\left(\frac{1+2\sqrt{2}+2}{4}\right) + \frac{1}{4} = 1 + \frac{\sqrt{2}}{2}$

$\frac{4}{4} + \frac{2\sqrt{2}}{4} = 1 + \frac{\sqrt{2}}{2}$

$1 + \frac{\sqrt{2}}{2} = 1 + \frac{\sqrt{2}}{2}$ ✓

11. "P" is a fixed point on the diameter $\overline{AB}$ of a circle. Prove that for any chord $\overline{CD}$ of the circle that is parallel to $\overline{AB}$, we have $PC^2 + PD^2 = PA^2 + PB^2$

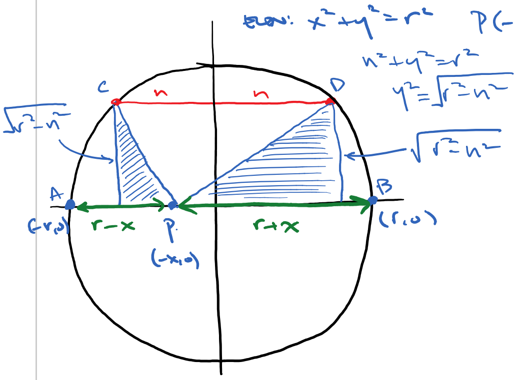
① IF A CHORD IS PARALLEL WITH THE DIAMETER THEN IT IS SYMMETRICAL WITH THE DIAMETER

② LET THE LENGTH OF CHORD CD BE '2n'

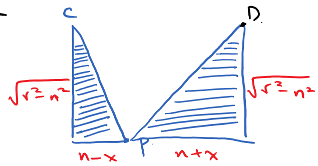
EQN: $x^2 + y^2 = r^2$ $\quad P(-x, 0)$

$n^2 + y^2 = r^2$

$y = \sqrt{r^2 - n^2}$



③ USE THE BLUE TRIANGLE



$PC^2 = (n-x)^2 + (\sqrt{r^2-n^2})^2$

$= \cancel{n^2} - 2nx + x^2 + r^2 - \cancel{n^2}$

$PD^2 = (n+x)^2 + (\sqrt{r^2-n^2})^2$

$= \cancel{n^2} + 2nx + x^2 + r^2 - \cancel{n^2}$

So $\boxed{PC^2 + PD^2 = 2x^2 + 2r^2}$

$PA^2 = (r-x)^2$

$= r^2 - 2rx + x^2$

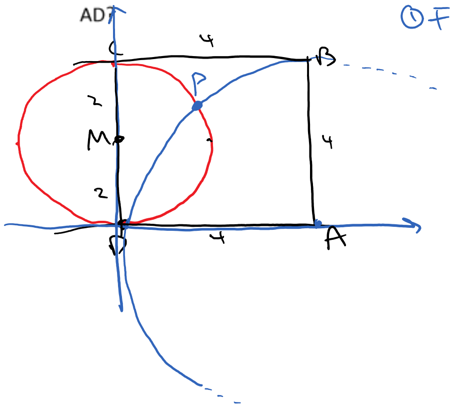
$PB^2 = (r+x)^2$

$= r^2 + 2rx + x^2$

$\boxed{PA^2 + PB^2 = 2r^2 + 2x^2}$

$\therefore \; PA^2 + PB^2 = PC^2 + PD^2$

12. Square ABCD has sides of length 4, and “M” is the midpoint of $\overline{CD}$. A circle with radius “2” and centre “M” intersects a circle with radius 4 and center “A” at points “P” and “D. What is the distance from “P” to $\overline{AD}$ AD?



① Find eqn of each circle:

$(x-4)^2+y^2=16$

$x^2+(y-2)^2=4$

$\cancel{x^2} - 8x + \cancel{16} + \cancel{y^2} = \cancel{16}$

$\cancel{x^2} + \cancel{y^2} - 4y + \cancel{4} = \cancel{4}$

$4y = 8x$

$\boxed{y = \frac{8x}{4} = 2x}$

$y^2 = 4x^2$

$\frac{y^2}{4} = x^2$

$x^2 + y^2 - 4y + 4 = 4$

$\frac{5y^2}{4} - 4y = 0$

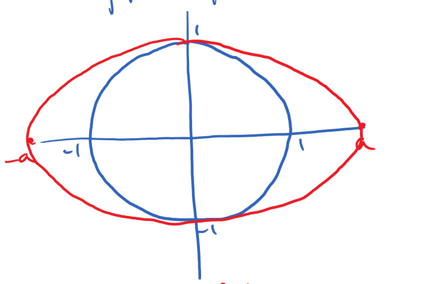
$5y^2 - 16y = 0$

$y(5y-16) = 0$

$\boxed{y = \frac{16}{5}}$

13. Scaling a closed figure in some direction by a factor “k” produces a figure with area “k” times the area of the original figure. Use this fact to explain why the area of an ellipse with major axis of length 2a and minor axis of length 2b is $\pi \times a \times b$

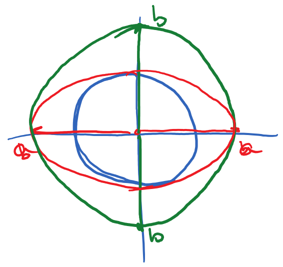
① Suppose you begin with a circle of radius 1.



$x \rightarrow ax.$

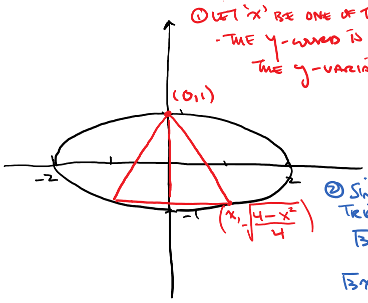
So Area $= \pi \longrightarrow$ Area $= a\pi.$

② Now expand the vertical axis by "b"



Area of new shape is $\boxed{\pi \times a \times b}$

14. An equilateral triangle is inscribed in the ellipse whose equation is $x^2 + 4y^2 = 4$. One vertex of the triangle is (0,1), and one altitude is contained in the y-axis. Find the length of each side of the triangle [AIME]



① LET 'x' BE ONE OF THE BOTTOM POINTS.
- THE y-COORD IS FOUND BY ISOLATING THE y-VARIABLE: $x^2+4y^2=4$

$4y^2 = 4-x^2$

$y = \sqrt{\frac{4-x^2}{4}}$

② SINCE IT'S AN EQUILATERAL TRIANGLE, THE HEIGHT IS $\sqrt{3}x$.

$\sqrt{3}x = 1 + \sqrt{\frac{4-x^2}{4}}$

$\sqrt{3}x - 1 = \sqrt{\frac{4-x^2}{4}}$

$(\sqrt{3}x-1)(\sqrt{3}x-1) = \frac{4-x^2}{4}$

$3x^2 - 2\sqrt{3}x + 1 = 1 - \frac{x^2}{4}$

$12x^2 + x^2 - 2\sqrt{3}x = 0$

$13x^2 - 2\sqrt{3}x = 0$

$x(13x - 2\sqrt{3}) = 0$

$\boxed{x = \frac{2\sqrt{3}}{13}}$ //

③ LENGTH OF SIDE OF TRIANGLE

$S = 2x$

$\boxed{S = \frac{4\sqrt{3}}{13}}$ //

15. Given that $x^2 + y^2 = 14x + 6y + 6$, what is the largest possible value that $3x + 4y$ can have? [AHSME]

① $x^2 - 14x + y^2 - 6y = 6$

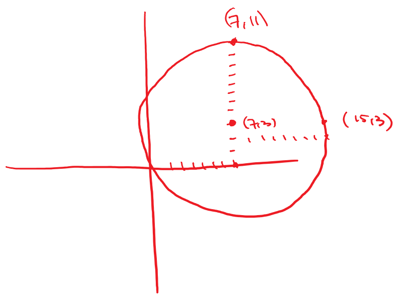
$x^2 - 14x + 49 + y^2 - 6y + 9 = 64$

$(x-7)^2 + (y-3)^2 = 64$

$(y-3)^2 = 64 - (x-7)^2$

$y - 3 = \sqrt{64-(x-7)^2}$

$y = 3 + \sqrt{64-(x-7)^2}$



② $Max = 3x + 4\left(3 + \sqrt{64-(x-7)^2}\right)$

$Max = 3x + 12 + 4\sqrt{64-(x-7)^2}$

$M = 3x + 12 + 4\left(64-(x-7)^2\right)^{1/2}$

$M' = 3 + 2\left(64-(x-7)^2\right)^{-1/2}\left(-2(x-7)\right)$

$0 = 3 + \frac{2(-2x+14)}{\sqrt{64-(x-7)^2}}$

$-3\sqrt{64-(x-7)^2} = -(4x+28)$

$9\left(64-(x-7)^2\right) = 16x^2 + 112x + 28^2$

$24^2 - 9(x^2-14x+49) = 16x^2 + 112x + 28^2$

$24^2 - 9x^2 + 126x - 21^2 = 16x^2 + 112x + 28^2$

$0 = 25x - 14x - 233$

5

5

16. Let $f(x)=(x+3)^2+\frac{9}{4}$ for $x\geq 3$. Find the shortest possible distance between a point on the graph of "f" and a point on the graph of f^{-1} [ARML]

$$\boxed{f(x) = (x+3)^2+\frac{9}{4}}$$

① Find $f^{-1}(x)$.

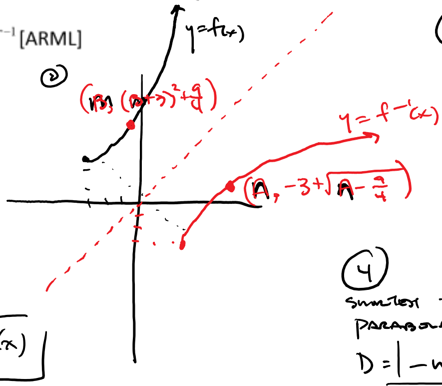
$$x = (y+3)^2+\frac{9}{4}$$

$$x-\frac{9}{4} = (y+3)^2$$

$$\sqrt{x-\frac{9}{4}} = y+3$$

$$\boxed{-3+\sqrt{x-\frac{9}{4}} = y = f^{-1}(x)}$$

②



③ Shortest Distance Formula

$$Ax+By+C=0$$

$$D=\frac{|Ax+By+C|}{\sqrt{A^2+B^2}}$$

i) $y=x$ $\quad \boxed{A=-1,\ B=1}$

$y-x=0$

ii) point $[m, (m+3)^2+\frac{9}{4}]$

④ Shortest Distance from parabola to $y=x$.

$$D=\frac{\left|-m+(m+3)^2+\frac{9}{4}\right|}{\sqrt{(-1)^2+(1)^2}}$$

$$D=\frac{\left|-m+m^2+6m+9+\frac{9}{4}\right|}{\sqrt{2}}$$

$$=\frac{\left|(m^2+5m)+9+\frac{9}{4}\right|}{\sqrt{2}}$$

$$=\frac{\left|\left(m^2+5m+\frac{25}{4}\right)-\frac{25}{4}+\frac{36}{4}+\frac{9}{4}\right|}{\sqrt{2}}$$

$$D=\frac{\left|\left(m+\frac{5}{2}\right)^2+\frac{20}{4}\right|}{\sqrt{2}}$$

Shortest distance will occur when $m=-\frac{5}{2}$ $\therefore$ $\boxed{D=\frac{20}{4\sqrt{2}}=\frac{5\sqrt{2}}{2}}$

⑤ Shortest distance b/n parabola & root function will be

$$\boxed{d=\frac{5\sqrt{2}}{2}\times 2 = 5\sqrt{2}}$$

* Note: The distance b/n their vertex:

$\left(-3, \frac{9}{4}\right)$ & $\left(\frac{9}{4}, -3\right)$

$$D^2 = \left(-3-\frac{9}{4}\right)^2+\left(\frac{9}{4}+3\right)^2$$

$$D = \sqrt{\left(\frac{21}{4}\right)^2+\left(\frac{21}{4}\right)^2} = \sqrt{2\left(\frac{21}{4}\right)^2}$$

$$\boxed{D=\frac{21}{4}(\sqrt{2})}$$

17. Determine the unique pair of real numbers (x,y) that satisfy the equation:

$$(4x^2+6x+4)(4y^2-12y+25)=28$$

$$(2x^2+3x+2)(4(y^2-3y)+25) = 14.$$

$$\left(2\left(x^2+\frac{3x}{2}+\frac{9}{16}-\frac{9}{16}\right)+2\right)\left(4\left(y^2-3y+\frac{9}{4}-\frac{9}{4}\right)+25\right) = 14.$$

$$\left(2\left(x+\frac{3}{4}\right)^2-\frac{9}{8}+2\right)\left(4\left(y-\frac{3}{2}\right)^2+16\right) = 14.$$

$$\left(\underbrace{2\left(x+\frac{3}{4}\right)^2+\frac{7}{8}}_{\text{Min}=\frac{7}{4}}\right)\left(\underbrace{4\left(y-\frac{3}{2}\right)^2+16}_{\text{Min}=16}\right) = 14$$

so this only occurs when $\boxed{x=-\frac{3}{4} \text{ \& } y=\frac{3}{2}}$

18. The graph of $2x^2 + xy + 3y^2 - 11x - 20y + 40 = 0$ is an ellipse in the first quadrant of the xy-plane. Let “a” and “b” be the maximum and minimum values of $\frac{y}{x}$ over all points (x, y) on the ellipse. What is the value of “a+b” [AMC]

① CONVERT EQN TO GENERAL FORM.

$2x^2 - 11x + 3y^2 - 20y + xy + 40 = 0$.